

# ISOMORPHIC COPIES OF $l^\infty$ IN CESÀRO-ORLICZ FUNCTION SPACES

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**ABSTRACT.** We characterize Cesàro-Orlicz function spaces  $Ces_\varphi$  containing isomorphic copy of  $l^\infty$ . We also describe the subspaces  $(Ces_\varphi)_a$  of all order continuous elements of  $Ces_\varphi$ . Finally, we study the monotonicity structure of the spaces  $Ces_\varphi$  and  $(Ces_\varphi)_a$ .

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## 1. INTRODUCTION

The structure of different types of Cesàro spaces has been widely investigated during the last decades from the isomorphic as well as isometric point of view. Firstly, the Cesàro sequence spaces  $ces_p$  has been studied by many authors starting from 1970. Then the attention has been moved to the Cesàro function spaces  $Ces_p$ . It was interesting among others that some properties are fulfilled in the sequence case and do not in the function case. Moreover, it happens that the cases  $Ces_p[0, 1]$  and  $Ces_p[0, \infty)$  are essentially different (see the description of the Köthe dual of Cesàro spaces in [1] and [19]). Probably the most important papers concerning the structure of the spaces  $Ces_p$  are two papers by Astashkin and Maligranda (see [1], [2]). Recall that the space  $Ces_p[0, 1]$ , for  $1 \leq p \leq \infty$ , consists of those Lebesgue measurable real functions  $f$  on  $[0, 1]$  for which the Cesàro-means  $C|f|(x) = \frac{1}{x} \int_0^x |f(t)|dt$  belong to  $L_p[0, 1]$  (similarly for the spaces  $ces_p$  and  $Ces_p[0, \infty)$ ). It is natural to investigate the space

$$CX = CX(I) = \{f \in L^0(I) : C|f| \in X\}$$

for any Banach ideal space  $X$  on  $I$ , where  $I = [0, 1]$  or  $I = [0, \infty)$ . These spaces have been defined in [29] for  $X$  being a Banach ideal space on  $[0, \infty)$  and, for example, in [11], [19] for  $X$  being a symmetric space on  $I$ . If we take instead of  $X$  the Orlicz sequence space  $l^\varphi$  (the generalization of  $l^p$ ) then the space  $CX$  becomes the Cesàro-Orlicz sequence space denoted by  $ces_\varphi$  which the structure is of course more rich than for the Cesàro sequence space  $ces_p$ . The spaces  $ces_\varphi$  has been studied by many authors (e.g. [8], [9], [10], [14] and [16]). If we put  $X = L^\varphi(I)$  the Orlicz function space then  $CX$  becomes the Cesàro-Orlicz function space  $Ces_\varphi(I)$ . As far as we know the structure of these spaces has not been investigated until now. We want to characterize these spaces  $Ces_\varphi$  which contain an order isomorphic copy of  $l^\infty$  (equivalently, are not order continuous). Clearly, such results are very useful in further studying isomorphic structure of these spaces. We characterize the subspaces  $(Ces_\varphi)_a$  of all order continuous elements

in  $Ces_\varphi$ . Finally, we prove criteria for strict monotonicity of the spaces  $(Ces_\varphi)_a$ . We also give a characterization of uniform monotonicity in Cesàro-Orlicz function spaces. We consider the largest possible class of Orlicz functions giving the maximal generality of spaces under consideration.

## 2. PRELIMINARIES

The symbols  $\mathbb{R}$ ,  $\mathbb{R}_+$  and  $\mathbb{N}$  denote the sets of reals, nonnegative reals and natural numbers, respectively. Let  $L^0 = L^0(I)$  be the space of all classes of real-valued Lebesgue measurable functions defined on  $I$ , where  $I = [0, 1]$  or  $I = [0, \infty)$ . A Banach space  $E = (E, \|\cdot\|)$  is said to be a Banach ideal space on  $I$  if  $E$  is a linear subspace of  $L^0(I)$  and satisfies two conditions:

- (i) if  $g \in E$ ,  $f \in L^0$  and  $|f| \leq |g|$  a.e. on  $I$  then  $f \in E$  and  $\|f\| \leq \|g\|$ ,
- (ii) there is an element  $f \in E$  that is positive on whole  $I$ .

Sometimes we write  $\|\cdot\|_E$  to be sure in which space the norm has been taken.

For two Banach ideal spaces  $E$  and  $F$  on  $I$  the symbol  $E \hookrightarrow F$  means that the embedding  $E \subset F$  is continuous, i.e., there exists constant a  $C > 0$  such that  $\|x\|_F \leq C \|x\|_E$  for all  $x \in E$ . Moreover,  $E = F$  means that the spaces are the same as the sets and the norms are equivalent.

Recall that for  $f \in E$  the distribution function  $d_f$  is defined by

$$d_f(\lambda) = m(\{t \in I : |f(t)| > \lambda\})$$

for all  $\lambda > 0$ , where  $m$  is the Lebesgue measure. The non-increasing rearrangement of  $f$  is denoted by  $f^*$  and is defined as

$$f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) < t\}$$

for  $t \geq 0$ . We say that two functions  $f, g \in L^0(I)$  are equimeasurable if they have the same distribution functions  $d_f \equiv d_g$ . Then we write  $f \sim g$ . By a symmetric function space (symmetric Banach function space or rearrangement invariant Banach function space) on  $I$  we mean a Banach ideal space  $(E, \|\cdot\|_E)$  with an additional property that for any two functions  $f \in E, g \in L^0(I)$ , with  $f \sim g$  we have  $g \in E$  and  $\|f\|_E = \|g\|_E$ . In particular,  $\|f\|_E = \|f^*\|_E$ .

A point  $f \in E$  is said to have an order continuous norm (or to be an order continuous element) if for each sequence  $(f_n) \subset E$  satisfying  $0 \leq f_n \leq |f|$  and  $f_n \rightarrow 0$  a.e. on  $I$ , one has  $\|f_n\| \rightarrow 0$ . By  $E_a$  we denote the subspace of all order continuous elements of  $E$ . It is worth to notice that in case of Banach ideal spaces on  $I$ ,  $x \in E_a$  if and only if  $\|x\chi_{A_n}\| \rightarrow 0$  for any decreasing sequence of Lebesgue measurable sets  $A_n \subset I$  with empty intersection. A Banach ideal space is called order continuous ( $E \in (OC)$  shortly) if every element of  $E$  is order continuous, i.e.,  $E = E_a$ .

We say a Banach ideal space  $X$  is strictly monotone ( $X \in (SM)$ ), if  $\|x\| < \|y\|$  for all  $x, y \in X$  such that  $0 \leq x \leq y$  and  $x \neq y$ . A Banach ideal space  $X$  is uniformly monotone ( $X \in (UM)$ ) if for each  $\epsilon \in (0, 1)$  there exists  $\delta(\epsilon) \in (0, 1)$  such that for all  $0 \leq y \leq x$ ,  $\|x\| = 1$  and  $\|y\| \geq \epsilon$ , we have  $\|x - y\| \leq 1 - \delta(\epsilon)$ .  $X$  is called lower locally uniformly monotone ( $X \in (LLUM)$ ) if for any  $x \in X$ ,  $\|x\| = 1$  and  $\epsilon \in (0, 1)$  there exists  $\delta(x, \epsilon) \in (0, 1)$  such that  $\|x - y\| \leq 1 - \delta(x, \epsilon)$  whenever  $0 \leq y \leq x$  and  $\|y\| \geq \epsilon$ . Note that monotone properties of Banach ideal spaces are useful in dominated best approximation problems, see [7] for further references.

The next well known theorem shows the relation between the order continuity of  $E$  and the existing of isomorphic copy of  $l^\infty$ .

**Theorem A.** (G. Ya. Lozanovskii, see [24]) A Banach ideal space  $E$  is order continuous if and only if  $E$  contains no isomorphic copy of  $l^\infty$ .

A function  $\varphi : [0, \infty) \rightarrow [0, \infty]$  is called an Orlicz function if:

- (i)  $\varphi$  is convex,
- (ii)  $\varphi$  is non-decreasing,

- (iii)  $\varphi(0) = 0$ ,
- (iv)  $\varphi$  is neither identically equal to zero nor infinity on  $(0, \infty)$ ,
- (v)  $\varphi$  is left continuous on  $(0, \infty)$ , i.e.,  $\lim_{u \rightarrow b_\varphi^-} \varphi(u) = \varphi(b_\varphi)$  if  $b_\varphi < \infty$ , where

$$b_\varphi = \sup\{u > 0 : \varphi(u) < \infty\}.$$

For more information about Orlicz functions see [6] and [18].

If we denote

$$a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\},$$

then  $0 \leq a_\varphi \leq b_\varphi \leq \infty$ ,  $a_\varphi < \infty$ ,  $b_\varphi > 0$ , since an Orlicz function is neither identically equal to zero nor infinity on  $(0, \infty)$ . The function  $\varphi$  is continuous and nondecreasing on  $[0, b_\varphi]$  and is strictly increasing on  $[a_\varphi, b_\varphi]$ . We use notations  $\varphi > 0$ ,  $\varphi < \infty$  when  $a_\varphi = 0$ ,  $b_\varphi < \infty$ , respectively.

We say an Orlicz function  $\varphi$  satisfies the condition  $\Delta_2$  for large arguments ( $\varphi \in \Delta_2(\infty)$  for short) if there exists  $K > 0$  and  $u_0 > 0$  such that  $\varphi(u_0) < \infty$  and

$$\varphi(2u) \leq K\varphi(u)$$

for all  $u \in [u_0, \infty)$ . Similarly, we can define the condition  $\Delta_2$  for small, with  $\varphi(u_0) > 0$  ( $\varphi \in \Delta_2(0)$ ) or for all arguments ( $\varphi \in \Delta_2(\mathbb{R}_+)$ ). These conditions play a crucial role in the theory of Orlicz spaces, see [6], [18], [26] and [28].

We will write  $\varphi \in \Delta_2$  in two cases:  $\varphi \in \Delta_2(\infty)$  if  $I = [0, 1]$  and  $\varphi \in \Delta_2(\mathbb{R}_+)$  if  $I = [0, \infty)$ .

The Orlicz function space  $L^\varphi = L^\varphi(I)$  generated by an Orlicz function  $\varphi$  is defined by

$$L^\varphi = \{f \in L^0(I) : I_\varphi(f/\lambda) < \infty \text{ for some } \lambda = \lambda(f) > 0\},$$

where  $I_\varphi(f) = \int_I \varphi(|f(t)|) dt$  is a convex modular (for the theory of Orlicz spaces and modular spaces see [26] and [28]). The space  $L^\varphi$  is a Banach ideal space with the Luxemburg-Nakano norm

$$\|f\|_\varphi = \inf\{\lambda : I_\varphi(f/\lambda) \leq 1\}.$$

It is well known that  $\|f\|_\varphi \leq 1$  if and only if  $I_\varphi(f) \leq 1$ . Moreover, the set

$$KL^\varphi = KL^\varphi(I) = \{f \in L^0(I) : I_\varphi(f) < \infty\},$$

will be called the Orlicz class.

The continuous Cesàro operator is defined for  $0 < x \in I$  as

$$Cf(x) = \frac{1}{x} \int_0^x f(t) dt.$$

For a Banach ideal space  $X$  on  $I$  we define an abstract Cesàro space  $CX = CX(I)$  as

$$CX = \{f \in L^0(I) : C|f| \in X\}$$

with a norm  $\|f\|_{CX} = \|C|f|\|_X$ .

The Cesàro-Orlicz function space  $Ces_\varphi = Ces_\varphi(I)$  is defined as  $Ces_\varphi(I) = CL^\varphi(I)$ . Consequently, the norm in the space  $Ces_\varphi$  is given by the formula

$$\|f\|_{Ces(\varphi)} = \inf\{\lambda : \rho_\varphi(f/\lambda) \leq 1\}$$

where  $\rho_\varphi(f) = I_\varphi(C|f|)$  is a convex modular.

The dilation operator  $D_s$ ,  $s > 0$ , defined on  $L^0(I)$  by

$$D_s x(t) = x(t/s) \chi_I(t/s) = x(t/s) \chi_{[0, \min\{1, s\})}(t),$$

for  $t \in I$ , is bounded in any symmetric space  $E$  on  $I$  and  $\|D_s\|_{E \rightarrow E} \leq \max\{1, s\}$  (see [3, p. 148]). Moreover, the lower and upper Boyd indices of  $E$  are defined by

$$p(E) = \lim_{s \rightarrow 0^+} \frac{\ln \|D_s\|_{E \rightarrow E}}{\ln s},$$

$$q(E) = \lim_{s \rightarrow \infty} \frac{\ln \|D_s\|_{E \rightarrow E}}{\ln s}.$$

In particular, they satisfy the inequalities

$$1 \leq p(E) \leq q(E) \leq \infty.$$

In the case when  $E$  is the Orlicz space  $L^\varphi$ , these indices correspond to the so-called Orlicz-Matuszewska indices of Orlicz functions generating the Orlicz spaces, i.e.,  $\alpha_\varphi = p(L^\varphi)$  and  $\beta_\varphi = q(L^\varphi)$ , where  $\alpha_\varphi$  and  $\beta_\varphi$  are the lower and upper Orlicz-Matuszewska indices of the Orlicz space  $L^\varphi$  (see [23, Proposition 2.b.5 and Remark 2 on page 140]). For more details see [4], [5], [25], [26] and [28].

In this paper we accept the convention that  $\sum_{n=m}^k x_n = 0$  if  $k < m$ .

Let us mention the important result about boundedness of the operator  $C$ .

**Theorem B.** [17, p. 127] For any symmetric space  $E$  on  $I$  the operator  $C : E \rightarrow E$  is bounded if and only if the lower Boyd index satisfies  $p(E) > 1$ .

The immediate consequence of Theorem B and the above discussion is a next corollary.

**Corollary 1.** The embedding  $L^\varphi \hookrightarrow Ces_\varphi$  holds if and only if  $\alpha_\varphi > 1$ .

**Remark.** Let  $X, Y$  be Banach ideal spaces on  $I$ . If  $X \hookrightarrow Y$  then  $CX \hookrightarrow CY$ . Indeed, suppose  $X \hookrightarrow Y$ . Then for all  $x \in CX$

$$\|x\|_{CY} = \|C|x|\|_Y \leq A \|C|x|\|_X = A \|x\|_{CX},$$

for some constant  $A > 0$ . This means that  $CX \hookrightarrow CY$ .

Let  $L^\varphi$  and  $L^\psi$  be the Orlicz spaces with  $\varphi, \psi < \infty$ . The criteria for the embeddings  $L^\varphi \hookrightarrow L^\psi$  can be found in [26, Theorem 3.4 p. 18]. Consequently, we conclude that:

- (i) if there exists  $k > 0$  with  $\psi(u) \leq \varphi(ku)$  for all  $u \in [0, \infty)$  then  $Ces_\varphi[0, \infty) \hookrightarrow Ces_\psi[0, \infty)$ ,
- (ii) if there exists  $k, u_0 > 0$  such that  $\psi(u) \leq \varphi(ku)$  for all  $u \in [u_0, \infty)$  then  $Ces_\varphi[0, 1] \hookrightarrow Ces_\psi[0, 1]$ ,
- (iii) if there exists  $k, u_0 > 0$  such that  $\psi(u) \leq \varphi(ku)$  for all  $u \in [0, u_0]$  then  $ces_\varphi \hookrightarrow ces_\psi$ .

Note that case (iii) is exactly Proposition 1 in [16] but now it is an immediate consequence of our above remark.

### 3. RESULT

**Fact.** Let  $X$  be a Banach ideal space on  $I$ . By the definition, the order continuity of  $X$  gives the same for  $CX$  (for proof see [19, Lemma 1]). The case of strict monotonicity and uniform monotonicity is similar. The converse is not true in general (see [11, Proposition 2.1] and [21, Example 1]).

*Proof.* We proof only the implication for uniform monotonicity. We apply Theorem 6 (ii) in [13]. Suppose that  $X \in (\text{UM})$ . Take  $\epsilon > 0$ ,  $x, y \in CX$ ,  $x, y \geq 0$ ,  $\|x\|_{CX} = 1$  and  $\|y\|_{CX} \geq \epsilon$ . We have

$$\|x + y\|_{CX} = \|C|x + y|\|_X = \|C|x| + C|y|\|_X.$$

Since  $X$  is uniformly monotone we have that there is a  $\sigma(\epsilon) > 0$  such that

$$\|C|x| + C|y|\|_X \geq 1 + \sigma(\epsilon).$$

But this means that  $CX \in (\text{UM})$ . □

**Proposition 2.** Suppose  $X$  is a symmetric space on  $I = [0, 1]$  and  $C : X \rightarrow X$ . Then  $X \in (\text{OC})$  if and only if  $CX \in (\text{OC})$ .

*Proof. Necessity.* It follows from Fact above.

*Sufficiency.* We thank Professor Karol Leśnik for giving the proof.

Suppose that  $X \notin (\text{OC})$ , i.e. there exists  $0 < f \in X \setminus X_a$ . Because  $X$  is symmetric we can assume that  $f = f^*$  (see [7, Lemma 2.6]). Then  $Cf \in X$  and  $Cf \geq f$ . Applying the symmetry of  $X$  and passing to subsequence, if necessary, we can assume that there is a  $\delta > 0$  such that

$$\|f\chi_{[0,1/n]}\|_X \geq \delta,$$

for all  $n \in \mathbb{N}$ . Since

$$C(f\chi_{[0,1/n]})(t) \geq f\chi_{[0,1/n]}(t),$$

for all  $t \in I$  and  $n \in \mathbb{N}$ , we have

$$\|f\chi_{[0,1/n]}\|_{CX} = \|C(f\chi_{[0,1/n]})\|_X \geq \|f\chi_{[0,1/n]}\|_X \geq \delta > 0,$$

which means that  $X \notin (\text{OC})$ . □

**Proposition 3.** Let  $\varphi$  be an Orlicz function. The following conditions are equivalent:

- (i) the space  $Ces_\varphi[0, \infty) \neq 0$ ,
- (ii) there exists  $\lambda_0 > 0$  and  $x_0 \in [0, \infty)$  such that  $\int_{x_0}^\infty \varphi(\lambda_0/t)dt < \infty$ ,
- (iii) for each  $\lambda_0 > 0$  there exist  $y_0 \in [0, \infty)$  with  $\int_{y_0}^\infty \varphi(\lambda_0/t)dt < \infty$ .

*Proof.* The equivalence of conditions (i) and (ii) follows from Theorem 1 (a) in [19]. Clearly, (iii)  $\Rightarrow$  (ii).

(ii)  $\Rightarrow$  (iii). We have to consider two cases. If  $\lambda \leq \lambda_0$ , there is nothing to prove. Now suppose  $\lambda > \lambda_0$ . Then

$$\int_{y_0}^\infty \varphi(\lambda/t)dt = \int_{y_0}^\infty \varphi\left(\frac{\lambda_0}{\lambda_0 t/\lambda}\right)dt = \frac{\lambda}{\lambda_0} \int_{\lambda_0 y_0/\lambda}^\infty \varphi(\lambda_0/u)du$$

therefore it is enough to take  $y_0 = (\lambda/\lambda_0)x_0$ . □

**Remark 3.1.** Let  $\varphi$  be an Orlicz function satisfying condition (S):

$$\liminf_{t \rightarrow 0} t\varphi'(t)/\varphi(t) > 1.$$

Then  $Ces_\varphi[0, \infty) \neq 0$ .

*Proof.* Firstly, condition (S) implies (ii) in Proposition 3. Indeed, we can use the same argument as in the proof of implication (a)  $\Rightarrow$  (c) in [8, Theorem 2.2]. But condition (ii) is equivalent to  $Ces_\varphi[0, \infty) \neq 0$ . □

**Remark 4.** Let  $\varphi$  be an Orlicz function. Then  $Ces_\varphi[0, 1] \neq 0$ .

*Proof.* It follows from Theorem 1 (b) in [19] that  $Ces_\varphi[0, 1] \neq 0$  if and only if there exists  $0 < a < 1$  such that  $\chi_{[a,1]} \in L^\varphi[0, 1]$ . But  $L^\varphi[0, 1]$  is a symmetric space and

$$L^\infty[0, 1] = L^1[0, 1] \cap L^\infty[0, 1] \hookrightarrow L^\varphi[0, 1] \hookrightarrow L^1[0, 1] + L^\infty[0, 1] = L^1[0, 1]$$

(see [3]). Therefore  $\chi_{[a,1]} \in L^\varphi[0, 1]$  for all  $a \in (0, 1)$ . □

The following theorem gives a similar information as in the sequence space  $ces_\varphi$  one can get from Theorem 2.3 in [8]. Before we formulate the theorem define a set

$$C_\varphi = C_\varphi(I) = \{x \in Ces_\varphi(I) : \rho_\varphi(kx) < \infty \text{ for all } k > 0\}.$$

Note that  $C_\varphi = \{0\}$  whenever  $b_\varphi < \infty$ .

**Theorem 5.** Suppose  $\varphi < \infty$ . Then the following conditions are true:

- (i)  $C_\varphi$  is the subspace of all order continuous elements of  $Ces_\varphi$ ,
- (ii)  $C_\varphi$  is a closed separable subspace of  $Ces_\varphi$ .

*Proof.* (i). We will show that  $C_\varphi \hookrightarrow (Ces_\varphi)_a$ . Take any  $x \in C_\varphi$  and a sequence  $(A_n)$  with  $A_n \searrow \emptyset$ . It is enough to show, that  $\|x\chi_{A_n}\|_{Ces(\varphi)} \rightarrow 0$  or equivalently  $\rho_\varphi(kx\chi_{A_n}) \rightarrow 0$  for any  $k > 0$ . Set  $k > 0$ . Clearly,

$$\lim_{n \rightarrow \infty} m((0, t] \cap A_n) = 0,$$

for each  $t > 0$ . Then for almost each  $t > 0$

$$0 \leq C(|x\chi_{A_n}|)(t) = \frac{1}{t} \int_0^t |x(s)\chi_{A_n}(s)| ds \rightarrow 0.$$

Moreover,

$$\rho_\varphi(kx\chi_{A_n}) \leq \rho_\varphi(kx) < \infty,$$

for each  $k$ . Thus, by the Lebesgue dominated convergence theorem, we have  $\rho_\varphi(kx\chi_{A_n}) \rightarrow 0$ , so  $x \in (Ces_\varphi(I))_a$ .

Next we prove the reverse inclusion. Define a set

$$B_\varphi = B_\varphi(I) = \text{cl}\{f \in Ces_\varphi(I) : f \text{ is simple function and } m(\text{supp}(f)) < \infty\}.$$

We claim that :

$$C_\varphi = B_\varphi. \quad (A)$$

Consider the inclusion  $B_\varphi \subset C_\varphi$ . Of course, if  $B_\varphi = \emptyset$  then  $B_\varphi \subset C_\varphi$  so we can assume that  $B_\varphi \neq \emptyset$ . We divide the proof in three cases.

- (1) Suppose  $x = \chi_{[a, b]}$ , where  $[a, b] \subset I$ ,  $0 \leq a < b < \infty$ . Since  $\varphi < \infty$  and  $Ces_\varphi \neq 0$ , by Proposition 3 we conclude that  $\rho_\varphi(kx) < \infty$  for all  $k > 0$ .
- (2) Let  $x = \sum_{k=1}^n c_k \chi_{A_k}$ ,  $A_k = [a_k, b_k] \subset I$  for  $n, k \in \mathbb{N}$  and  $0 \leq a_k < b_k < \infty$ . Using the convexity of modular  $\rho_\varphi$  and argument from case (1), we obtain  $\rho_\varphi(\lambda x) < \infty$  for all  $\lambda > 0$ .
- (3) Assume that there exists a sequence  $(x_n)$  such that  $x_n \rightarrow x$  in  $Ces_\varphi$  and  $x_n$  are of the form (2). Let  $k > 0$ . For sufficiency large  $n \in \mathbb{N}$  we have

$$\begin{aligned} \rho_\varphi(kx) &= \rho_\varphi(k(x - x_n + x_n)) = \rho_\varphi\left(\frac{1}{2}2k(x - x_n) + \frac{1}{2}2kx_n\right) \\ &\leq \frac{1}{2}\rho_\varphi(2k(x - x_n)) + \frac{1}{2}\rho_\varphi(2kx_n) \leq 1 + \rho_\varphi(2kx_n) < \infty. \end{aligned}$$

Therefore  $B_\varphi \subset C_\varphi$ .

Now, we will prove the reverse inclusion  $C_\varphi \subset B_\varphi$ . Let  $x \in C_\varphi$ ,  $k > 0$  and  $\epsilon > 0$ . Since  $\int_I \varphi(C(k|x|))dt < \infty$  there exists  $\beta = \beta(k) \in \mathbb{R}_+$  with

$$\int_\beta^\infty \varphi(C(2k|x(t)|))dt < \epsilon/2.$$

Put  $z_n = \varphi(C(2k|x|\chi_{[0, 1/n]}))$  and  $z = \varphi(C(2k|x|))$ . Then  $z \in L^1$ ,  $0 \leq z_n \leq z$  and  $z_n \rightarrow 0$  a.e. on  $I$ . Note that  $L^1 \in (\text{OC})$  whence  $\|z_n\|_{L^1} \rightarrow 0$ . Consequently, there exists  $\alpha = \alpha(k) \in \mathbb{R}_+$  such that

$$\int_\beta^\infty \varphi(C(2k|x|\chi_{[0, \alpha]}))dt < \epsilon/2.$$

Define an element  $x_n = x\chi_{[1/n, n]}$  for  $n \in \mathbb{N}$ . For sufficiently large  $n \in \mathbb{N}$  satisfying  $1/n < \alpha$  and  $n > \beta$  we have

$$\rho_\varphi(k(x_n - x)) = \rho_\varphi(kx\chi_{[0, 1/n) \cup (n, \infty)}) = \int_0^\infty \varphi(C(kx(t)\chi_{[0, 1/n) \cup (n, \infty)}(t)))dt$$

$$\begin{aligned}
&= \int_0^\infty \varphi(C(kx(t)\chi_{[0,1/n)}(t)) + C(kx(t)\chi_{(n,\infty)}(t)))dt \\
&\leq \frac{1}{2} \int_0^\infty \varphi(C(kx(t)\chi_{[0,1/n)}(t)))dt + \frac{1}{2} \int_0^\infty \varphi(C(kx(t)\chi_{(n,\infty)}(t)))dt \\
&\leq \frac{1}{2} \int_0^\infty \varphi(C(kx(t)\chi_{[0,\alpha)}(t)))dt + \frac{1}{2} \int_0^\infty \varphi(C(kx(t)\chi_{(\beta,\infty)}(t)))dt < \epsilon
\end{aligned}$$

i.e.  $\|x_n - x\|_{Ces(\varphi)} \rightarrow 0$ . Now, it is enough to prove that for each interval  $[a, b]$ ,  $0 \leq a < b < \infty$ , there is a sequence of functions  $y_n$  of the form (2) with

$$\|x\chi_{[a,b]} - y_n\|_{Ces(\varphi)} \rightarrow 0,$$

as  $n \rightarrow 0$ . Without loss of generality we may assume that  $x\chi_{[a,b]} \geq 0$ . Since  $x \in L^0$  we can find a sequence  $(y_n)$  of simple functions such that  $y_n \nearrow x\chi_{[a,b]}$ . Thus

$$0 \leq x\chi_{[a,b]} - y_n \leq x\chi_{[a,b]},$$

and  $x\chi_{[a,b]} - y_n \rightarrow 0$  a.e. on  $I$ . But  $x \in C_\varphi \subset (Ces_\varphi(I))_a$ , whence

$$\|x\chi_{[a,b]} - y_n\|_{Ces(\varphi)} \rightarrow 0.$$

This proves the claim (A).

Take  $x \in (Ces_\varphi(I))_a$ . Define an element  $x_n = x\chi_{[1/n,n]}$ . Then  $\|x_n - x\|_{Ces(\varphi)} \rightarrow 0$ . Moreover,  $x_n \in B_\varphi$  for all  $n \in \mathbb{N}$  by the above proof. Since  $B_\varphi$  is closed, so  $x \in B_\varphi = C_\varphi$ .

(ii) Clearly, since  $\rho_\varphi$  is a convex modular,  $C_\varphi$  is a linear subspace of  $Ces_\varphi$ .  $C_\varphi$  is closed, since  $B_\varphi = C_\varphi$  and  $B_\varphi$  is closed by the definition (note that  $X_a$  is closed for each Banach ideal space  $X$ , see [3, proof of Theorem 3.8, p. 16]). Separability of the space  $C_\varphi$  follows from (i) and the fact, that Lebesgue measure is separable, see [3, Theorem 5.5, p. 27].  $\square$

**Remark 6.** Note that from Theorem 3.11, p. 18 in [3] we have the following inclusions

$$(Ces_\varphi)_a \subset (Ces_\varphi)_b \subset Ces_\varphi.$$

But  $B_\varphi = (C_\varphi)_b$  (compare with the Definition 3.9, p. 17 in [3]) and  $B_\varphi \subset C_\varphi \subset (Ces_\varphi)_a$  from some parts of the proof of Theorem 5. Therefore, by different argumentation we also get  $C_\varphi = (Ces_\varphi)_a$ .

**Theorem 7.** Let  $\varphi$  be an Orlicz function and suppose  $C : L^\varphi \rightarrow L^\varphi$ . Then the following conditions are equivalent:

- (i)  $\varphi \in \Delta_2$ ,
- (ii)  $Ces_\varphi \in (OC)$ ,
- (iii)  $Ces_\varphi$  contains no isomorphic copy of  $l^\infty$ .

*Proof.* Equivalence (ii)  $\Leftrightarrow$  (iii) follows from Theorem A. Now we will proof the equivalence of conditions (i) and (ii).

(i)  $\Rightarrow$  (ii). If  $\varphi \in \Delta_2$ , then  $L^\varphi$  is order continuous, see e.g. [26, p. 21-22]. Therefore also  $Ces_\varphi$  is order continuous by the Fact.

(ii)  $\Rightarrow$  (i). We have to consider two cases.

I. Suppose  $I = [0, \infty)$ .

(1). Let  $b_\varphi < \infty$  and  $\varphi(b_\varphi) = \infty$ . Let  $(u_n) \subset \mathbb{R}_+$  be the sequence with  $u_n \nearrow b_\varphi^-$ . Of course,  $\varphi(u_n) \rightarrow \infty$ . Passing to a subsequence if necessary, we can assume that  $\varphi(u_n) \geq 1$  for all  $n \in \mathbb{N}$ . Let  $a_n = 1/2^n \varphi(u_n)$  for  $n \in \mathbb{N}$  and denote  $a = \sum_{n=1}^\infty a_n$ . Define a sequence of pairwise disjoint open intervals  $(A_n)_{n \in \mathbb{N}} \subset [0, 1]$  as follows

$$A_n = \left( a - \sum_{k=1}^n \frac{1}{2^k \varphi(u_k)}, a - \sum_{k=1}^{n-1} \frac{1}{2^k \varphi(u_k)} \right),$$

for all  $n \in \mathbb{N}$ . Then  $m(A_n) = a_n$  for all  $n \in \mathbb{N}$ . Let

$$x = \sum_{n=1}^{\infty} u_n \chi_{A_n}.$$

We claim that  $x \in Ces_{\varphi}$ . In fact,

$$\begin{aligned} I_{\varphi}(x) &= \int_0^{\infty} \varphi(x) dm = \int_0^{\infty} \sum_{n=1}^{\infty} \varphi(u_n) \chi_{A_n} dm \\ &= \sum_{n=1}^{\infty} \int_{A_n} \varphi(u_n) dm = \sum_{n=1}^{\infty} \varphi(u_n) m(A_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \end{aligned}$$

The assumption  $C : L^{\varphi} \rightarrow L^{\varphi}$  implies that  $L^{\varphi} \hookrightarrow Ces_{\varphi}$ , so  $x \in Ces_{\varphi}$ . Note that there exists  $N \in \mathbb{N}$  such that for all  $k \geq N$ ,  $2u_k > b_{\varphi}$ . Consequently, since  $x$  is non-increasing

$$\rho_{\varphi}(2x) = I_{\varphi}(C(2x)) = I_{\varphi}(2Cx) \geq I_{\varphi}(2x) = \sum_{n=1}^{\infty} \varphi(2u_n) m(A_n) = \infty,$$

and, by Theorem 5 (i), we conclude that  $Ces_{\varphi} \notin (OC)$ .

(2). Assume that  $b_{\varphi} < \infty$  and  $\varphi(b_{\varphi}) < \infty$ . Define an element  $x = b_{\varphi} \chi_{[0,1]}$ . Then

$$Cx(t) = b_{\varphi} \chi_{[0,1]}(t) + \frac{b_{\varphi}}{t} \chi_{(1,\infty)}(t),$$

and

$$\rho_{\varphi}(x) = I_{\varphi}(Cx) = \varphi(b_{\varphi}) + \int_1^{\infty} \varphi\left(\frac{b_{\varphi}}{t}\right) dt.$$

Since  $I_{\varphi}(x) < \infty$  so  $x \in L^{\varphi}$  and  $x \in Ces_{\varphi}$  because  $C : L^{\varphi} \rightarrow L^{\varphi}$ . Moreover,

$$\rho_{\varphi}(2x) = I_{\varphi}(C(2x)) \geq I_{\varphi}(2x) = \varphi(2b_{\varphi}) = \infty,$$

thus,  $Ces_{\varphi}[0, \infty) \notin (OC)$ , by Theorem 5 (i).

(3) Let  $a_{\varphi} > 0$ . Put  $x = a_{\varphi} \chi_{[0,\infty)}$ . Then  $x \in Ces_{\varphi}[0, \infty)$  and

$$\rho_{\varphi}(2x) = I_{\varphi}(C(2x)) = I_{\varphi}(2x) = \varphi(2a_{\varphi}) m([0, \infty)) = \infty.$$

Therefore,  $Ces_{\varphi}[0, \infty) \notin (OC)$  by Theorem 5 again.

(4). Now we assume that  $\varphi > 0$ ,  $\varphi < \infty$  and  $\varphi \notin \Delta_2(\mathbb{R}_+)$ . This means, that  $\varphi \notin \Delta_2(0)$  or  $\varphi \notin \Delta_2(\infty)$ . If  $\varphi \notin \Delta_2(\infty)$  then

$$\varphi(2u_n) \geq 2^n \varphi(u_n),$$

for some sequence  $(u_n) \nearrow \infty$ . Therefore in this case we can use the arguments from I (1) above. Now assume that  $\varphi \notin \Delta_2(0)$ . That means, that there exist a decreasing sequence  $(u_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ ,  $u_n \searrow 0$  and  $\varphi(2u_n) \geq 2^n \varphi(u_n)$  for all  $n \in \mathbb{N}$ . Define an element

$$x = \sum_{n=1}^{\infty} u_n \chi_{B_n},$$

where  $B_n = \left( \sum_{k=1}^{n-1} \frac{1}{2^k \varphi(u_k)}, \sum_{k=1}^n \frac{1}{2^k \varphi(u_k)} \right)$ . Then

$$\begin{aligned} I_{\varphi}(x) &= \int_0^{\infty} \varphi(x) dm = \int_0^{\infty} \sum_{n=1}^{\infty} \varphi(u_n) \chi_{B_n} dm \\ &= \sum_{n=1}^{\infty} \int_{B_n} \varphi(u_n) dm = \sum_{n=1}^{\infty} \varphi(u_n) m(B_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1, \end{aligned}$$

so  $x \in Ces_{\varphi}$  by the assumption  $C : L^{\varphi} \rightarrow L^{\varphi}$ . Furthermore, since  $x$  is non-increasing,

$$\rho_{\varphi}(2x) = I_{\varphi}(C(2x)) \geq I_{\varphi}(2x)$$



$$= \sum_{n=1}^{\infty} \varphi(2u_n)m(B_n) \geq \sum_{n=1}^{\infty} 2^n \varphi(u_n)m(B_n) = \sum_{n=1}^{\infty} 1 = \infty,$$

which means that the space  $Ces_{\varphi}[0, \infty)$  is not order continuous by Theorem 5 above.

II. Suppose  $I = [0, 1]$ .

(1). Let  $b_{\varphi} < \infty$  and  $\varphi(b_{\varphi}) = \infty$ . In this case we can use construction from case I (1) to show that  $Ces_{\varphi} \in (OC)$ .

(2). Assume that  $b_{\varphi} < \infty$  and  $\varphi(b_{\varphi}) < \infty$ . Now we follow as in the proof of case I (2) but in this case we don't need the assumption that  $C : L^{\varphi} \rightarrow L^{\varphi}$ . In fact, put  $x = b_{\varphi}\chi_{[0,1]}$ . Then

$$\rho_{\varphi}(x) = I_{\varphi}(Cx) = I_{\varphi}(x) = \int_0^1 \varphi(b_{\varphi})dm = \varphi(b_{\varphi}) < \infty,$$

so  $x \in Ces_{\varphi}$ . Furthermore,

$$\rho_{\varphi}(2x) = I_{\varphi}(C(2x)) = I_{\varphi}(2C(x)) = I_{\varphi}(2x) = \varphi(2b_{\varphi}) = \infty,$$

which means that  $Ces_{\varphi} \notin (OC)$ .

(3) Now suppose  $\varphi < \infty$  and  $\varphi \notin \Delta_2(\infty)$ . It is well known that  $L^{\varphi}[0, 1] \in (OC)$  if and only if  $\varphi \in \Delta_2(\infty)$  (see, e.g. [26, p. 21-22]). Therefore, from Proposition 2,  $Ces_{\varphi}[0, 1] \notin (OC)$ . Additionally, the same direct proof of case I (4) above works also for  $I = [0, 1]$ . □

An immediate consequence of Fact, Theorem 5 and Theorem 7 is the next corollary.

**Corollary 8.** Let  $\varphi$  be an Orlicz function.

- (i) If  $\varphi \in \Delta_2$ , then  $Ces_{\varphi} = (Ces_{\varphi})_a = C_{\varphi}$ .
- (ii) If  $C : L^{\varphi} \rightarrow L^{\varphi}$  then  $C_{\varphi} = Ces_{\varphi}$  if and only if  $\varphi \in \Delta_2$ .

**Lemma 9.** If  $x \in C_{\varphi}$ , then  $\|x\|_{Ces(\varphi)} = 1$  if and only if  $\rho_{\varphi}(x) = 1$ .

*Proof.* See the proof of Lemma 2.1 in [8]. □

**Theorem 10.** Let  $\varphi$  be an Orlicz function satisfying  $\varphi < \infty$ .

- (i) If  $I = [0, \infty)$ , then the space  $C_{\varphi}$  is strictly monotone if and only if  $\varphi > 0$ .
- (ii) If  $I = [0, 1]$  and  $\lim_{u \rightarrow \infty} \varphi(u)/u = \infty$ , then the space  $C_{\varphi}$  is strictly monotone if and only if  $\varphi > 0$ .

*Proof. Necessity.* Assume that  $a_{\varphi} > 0$ . We have to consider two cases:

- (i). If  $I = [0, \infty)$  then take  $y_t = t\chi_{[0,1]}$  for  $t \in \mathbb{R}_+$ . We have

$$(Cy_t)(u) = t\chi_{[0,1]}(u) + \frac{t}{u}\chi_{[1,\infty)}(u).$$

We define the function

$$f(t) = \int_0^{\infty} \varphi \left( t\chi_{[0,1]}(u) + \frac{t}{u}\chi_{[1,\infty)}(u) \right) du = \rho_{\varphi}(y_t),$$

for  $t > 0$ . Since  $t/x \rightarrow 0$  as  $x \rightarrow \infty$  and  $a_{\varphi} > 0$ , so there exists  $x_0 = x_0(t) \in \mathbb{R}_+$  such that  $t/x_0 \leq a_{\varphi}$ . Consequently,  $f(t) \leq \varphi(t)x_0$ , which means that  $f(t)$  takes finite values for  $t > 0$ . Moreover,  $f$  is convex function, so  $f$  is continuous function on  $\mathbb{R}_+$ . This means that  $f[\mathbb{R}_+] = \mathbb{R}_+$ . In that case, from the Darboux property we find  $\lambda > 0$  satisfying  $f(\lambda) = 1$ . Take  $x_1 > 1$  with  $\lambda/x_1 \leq a_{\varphi}/2$ . Let  $z = a_{\varphi}\chi_{[x_1, x_1+1]}/2$ . Then for any  $x \geq x_1$  we have

$$C(y_{\lambda} + z)(x) = Cy_{\lambda}(x) + Cz(x) \leq a_{\varphi},$$

whence  $\rho_\varphi(y_\lambda + z) = 1$  and, from Lemma 9, we get  $\|y_\lambda + z\|_{Ces(\varphi)} = 1$ . Summing up, we built elements  $u = y_\lambda$  i  $v = y_\lambda + z$  such that  $u \neq v$ ,  $u \leq v$  i  $\|u\|_{Ces(\varphi)} = \|v\|_{Ces(\varphi)} = 1$ . Moreover,  $u, v \in C_\varphi$ . This follows easily from the equality  $C_\varphi = B_\varphi$  (see proof of Theorem 5). This means, that the space  $C_\varphi$  isn't strictly monotone.

(ii). Suppose  $I = [0, 1]$ . For  $a, b \in \mathbb{R}$  set  $x = b\chi_{[0,a)}$ . Then

$$(Cx)(t) = b\chi_{[0,a)}(t) + \frac{ab}{t}\chi_{[a,1)}(t).$$

Since  $\lim_{u \rightarrow \infty} \varphi(u)/u = \infty$ , there is a number  $b_0 \in \mathbb{R}$  with  $b_0 > a_\varphi$  and  $a_\varphi\varphi(b_0)/b_0 > 1$ . We define the function

$$f(a) = a\varphi(b_0) + \int_a^{\frac{ab_0}{a_\varphi}} \varphi\left(\frac{ab_0}{t}\right) dt,$$

for  $a \in [0, a_\varphi/b_0]$ . Clearly,  $f(0) = 0$  and  $f(a_\varphi/b_0) > 1$ . Moreover, we claim that  $f$  is continuous on  $[0, a_\varphi/b_0]$ . Take  $a_n \rightarrow a$ ,  $a \in (0, a_\varphi/b_0]$  and put

$$A = \left\{t \in \left[0, \frac{a_\varphi}{b_0}\right] : a \leq t \leq \frac{ab_0}{a_\varphi}\right\},$$

$$A_n = \left\{t \in \left[0, \frac{a_\varphi}{b_0}\right] : a_n \leq t \leq \frac{a_nb_0}{a_\varphi}\right\},$$

for  $n \in \mathbb{N}$ . Then

$$\begin{aligned} 0 &\leq |f(a_n) - f(a)| \leq |a_n - a|\varphi(b_0) + \left| \int_a^{\frac{ab_0}{a_\varphi}} \varphi\left(\frac{ab_0}{t}\right) dt - \int_{a_n}^{\frac{a_nb_0}{a_\varphi}} \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \\ &= |a_n - a|\varphi(b_0) + \left| \int_{A \setminus A_n} \varphi\left(\frac{ab_0}{t}\right) dt + \int_{A \cap A_n} \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) dt - \int_{A_n \setminus A} \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \\ &\leq |a_n - a|\varphi(b_0) + \left| \int_{A \setminus A_n} \varphi\left(\frac{ab_0}{t}\right) dt \right| + \left| \int_{A \cap A_n} \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) dt \right| + \left| \int_{A_n \setminus A} \varphi\left(\frac{a_nb_0}{t}\right) dt \right|. \end{aligned}$$

Now we have

$$0 \leq \left| \int_{A \setminus A_n} \varphi\left(\frac{ab_0}{t}\right) dt \right| \leq m(A \setminus A_n) \sup_{t \in A} \varphi\left(\frac{ab_0}{t}\right) = m(A \setminus A_n) \varphi(b_0) \rightarrow 0,$$

as  $n \rightarrow \infty$ . Similarly, we can show that  $\left| \int_{A_n \setminus A} \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover, for  $n$  large enough  $a/2 < a_n < 2a$ . Therefore, for  $t \in A$

$$\left| \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) \right| \leq \max_{t_1, t_2 \in [a_\varphi, 2b_0]} |\varphi(t_1) - \varphi(t_2)| = \eta(a, b_0) = \eta < \infty,$$

because  $\varphi$  is continuous on the compact set  $[a_\varphi, 2b_0]$ . Since  $|\varphi(\frac{ab_0}{t}) - \varphi(\frac{a_nb_0}{t})| \rightarrow 0$  pointwise on  $A$ ,  $\eta\chi_A$  is integrable majorant and  $L^1[0, 1]$  is order continuous, so from Dominated Convergence Theorem we get

$$0 \leq \left| \int_{A \cap A_n} \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \leq \left| \int_A \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \rightarrow 0,$$

as  $n \rightarrow \infty$ . This means that  $|f(a_n) - f(a)| \rightarrow 0$ , so  $f$  is continuous. This proves the claim.

By the Darboux property of  $f$ , there exists  $a_1 \in (0, a_\varphi/b_0)$  such that  $f(a_1) = 1$ . Consider two elements

$$x_1 = b_0\chi_{[0,a_1)},$$

and

$$x_2 = b_0 \chi_{[0, a_1]} + \left( a_\varphi - \frac{a_1 b_0}{\delta} \right) \chi_{[\frac{\delta+1}{2}, 1]},$$

where  $\delta > 0$  is such that  $a_1 b_0 / a_\varphi < \delta < 1$ . Obviously,  $x_1 \leq x_2$  and  $x_1 \neq x_2$  and  $x_1, x_2 \in C_\varphi$ . Furthermore,  $\rho_\varphi(x_1) = \rho_\varphi(x_2) = 1$ . Indeed, we have the following inequalities

$$a_1 < \frac{b_0}{a_\varphi} a_1 < \delta < \frac{\delta+1}{2} < 1,$$

so for  $t \in [(\delta+1)/2, 1)$

$$(Cx_2)(t) = \frac{a_1 b_0}{t} + \left( a_\varphi - \frac{a_1 b_0}{\delta} \right) \frac{1}{t} \int_{\frac{\delta+1}{2}}^t ds < \frac{a_1 b_0}{t} + a_\varphi - \frac{a_1 b_0}{\delta} \leq a_\varphi.$$

Thus, by the Lemma 9,  $\|x_1\|_{Ces(\varphi)} = \|x_2\|_{Ces(\varphi)} = 1$ , whence the space  $C_\varphi$  is not strictly monotone.

*Sufficiency.* Assume, that  $a_\varphi = 0$ ,  $0 \leq x \leq y$ ,  $x \neq y$  and  $x, y \in C_\varphi$ . Without loss of generality, we can assume, that  $\|x\|_{Ces(\varphi)} = 1$ . From Lemma 9,  $\rho_\varphi(x) = 1$ . We need only to show that  $\rho_\varphi(y) > 1$ . Since  $\varphi$  is superadditive, so for any  $x, y \in (C_\varphi)_+$  we have

$$\rho_\varphi(x+y) \geq \rho_\varphi(x) + \rho_\varphi(y).$$

Since  $y - x \geq 0$ ,  $y - x \neq 0$  and  $\varphi > 0$ , we have  $\rho_\varphi(y - x) > 0$ , and consequently

$$\rho_\varphi(y) = \rho_\varphi(x + (y - x)) \geq \rho_\varphi(x) + \rho_\varphi(y - x) = 1 + \rho_\varphi(y - x) > 1.$$

□

**Theorem 11.** Let  $\varphi$  be an Orlicz function with  $\varphi < \infty$  and  $C : L^\varphi \rightarrow L^\varphi$ . Suppose additionally  $\lim_{u \rightarrow \infty} \varphi(u)/u = \infty$  if  $I = [0, 1]$ . The following conditions are equivalent:

- (i)  $Ces_\varphi \in (\text{UM})$ ,
- (ii)  $Ces_\varphi \in (\text{LLUM})$ ,
- (iii)  $\varphi > 0$  and  $\varphi \in \Delta_2$ .

*Proof.* (i)  $\Rightarrow$  (ii) by definition.

(ii)  $\Rightarrow$  (iii). If  $a_\varphi > 0$  then  $C_\varphi \notin (\text{SM})$  by Theorem 10. Moreover, see [12, Theorem 2.1], the following implication is true: if  $X \in (H_I)$  then  $X \in (\text{OC})$ . We can use the same proof to show that if  $X \in (\text{LLUM})$  then  $X \in (\text{OC})$ . Therefore, if  $\varphi \notin \Delta_2$  then  $Ces_\varphi \notin (\text{OC})$  by Theorem 7, whence  $Ces_\varphi \notin (\text{LLUM})$ .

(iii)  $\Rightarrow$  (i). If  $\varphi > 0$  and  $\varphi \in \Delta_2$  then  $L_\varphi \in (\text{UM})$ , see e.g. [13, Theorem 7]. Consequently  $Ces_\varphi \in (\text{UM})$  by Fact.

□

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